2.2 Semantics of Equations

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Now we want to define when an equation is "true" and when an equation follows from a set of equations. To this end, we use <u>interpretations</u> which give a meaning to function symbols, variables, and terms.

<u>Def. 2.2.1.</u> (Interpretation, Algebra)

For a signature Σ, a Σ-interpretation is a 3-tuple $I = (\mathcal{A}, \alpha, \beta)$.

$\mathcal{A}$ is the <u>carrier</u> of the interpretation. It is an arbitrary non-empty set.

$\alpha = (\alpha_f)_{f \in \Sigma}$ where $\alpha_f : \overbrace{\mathcal{A} \times \ldots \times \mathcal{A}}^{n \text{ times}} \to \mathcal{A}$ for $f \in \Sigma_n$. The function α_f is called the <u>meaning</u> of the fct. symbol f.

$\beta : \mathcal{V} \to \mathcal{A}$ is a <u>variable assignment</u>. It maps every variable x to an element $\beta(x)$ of the carrier $\mathcal{A}$.

<u>Ex. 2.2.2.</u> $I = (\mathcal{A}, \alpha, \beta)$ where

$\mathcal{A} = \mathbb{N} = \{0, 1, 2, \ldots\}$

$\alpha_{\mathcal{O}} = 0$

$\alpha_{succ}(n) = n + 1$

$\alpha_{plus}(n, m) = n + m$

$\alpha_{times}(n, m) = n \cdot m$

$\beta(x) = 5$

$\beta(y) = 3$

$I(plus(succ(x), y)) =$

$\alpha_{plus}(I(succ(x)), I(y)) =$

$\alpha_{plus}(\alpha_{succ}(I(x)), I(y)) =$

$\alpha_{plus}(\underbrace{\alpha_{succ}(\underbrace{\beta(x)}_{5})}_{6}, \underbrace{\beta(y)}_{3}) = 9$

<u>Def 2.2.1 (cont.)</u>

For every term t, the interpretation I now defines its meaning $I(t)$:

$$I(x) = \beta(x) \quad \text{for all } x \in \mathcal{V}$$

$$I(f(t_1, \ldots, t_n)) = \alpha_f(I(t_1), \ldots, I(t_n))$$

A Σ-interpretation without variable assignment is called a Σ-algebra $A = (\mathcal{A}, \alpha)$.

Ex 222 (cont.)

I' is like I, but with a different variable assignment:

$I' = (\underbrace{\mathcal{A}}_{\mathbb{N}}, \alpha, \beta')$ with $\beta'(x) = 2$, $\beta'(y) = 1$.

$$I(plus(succ(x), y)) = 9$$

$$I'(plus(succ(x), y)) = \alpha_{plus}(\alpha_{succ}(\beta'(x)), \beta'(y))$$

$$= 3 + 1 = 4$$

$I'' = (\mathbb{Q}, \alpha'', \beta)$ with

$\alpha''_0 = 0$, $\alpha''_{succ}(x) = x+1$, $\alpha''_{times}(x,y) = x \cdot y$, $\alpha''_{plus}(x,y) = \frac{x}{y}$ if $y \neq 0$,

$\alpha''_{plus}(x, 0) = 0$

$$I''(plus(succ(x), y)) =$$

$$\alpha''_{plus}(\underbrace{\alpha''_{succ}(\underbrace{\beta(x)}_{5})}_{6}, \underbrace{\beta(y)}_{3}) = \frac{6}{3} = 2$$

Def 223 (Satisfiability of Equations, Model)

An interpretation $I = (\mathcal{A}, \alpha, \beta)$ satisfies an equation $t_1 \equiv t_2$ iff $I(t_1) = I(t_2)$ (denoted "$I \models t_1 \equiv t_2$").

An algebra $A = (\mathcal{A}, \alpha)$ satisfies an equation $t_1 \equiv t_2$ (denoted "$A \models t_1 \equiv t_2$") iff $I \models t_1 \equiv t_2$ holds for all interpretations of the form $I = (\mathcal{A}, \alpha, \beta)$. So $t_1 \equiv t_2$ must hold for all variable assignments β, i.e., variables are implicitly universally quantified. We also say that A is a model of $t_1 \equiv t_2$.

Similarly, A is a model of a set of equations $\mathcal{E}$ (denoted "$A \models \mathcal{E}$") iff $A \models t_1 \equiv t_2$ for all $t_1 \equiv t_2 \in \mathcal{E}$.

Ex 224 $I = (\mathbb{N}, \alpha, \beta)$ with $\alpha_0 = 0$, $\alpha_{succ}(n) = n+1$, $\alpha_{plus}(n,m) = n+m$

$A = (\mathbb{N}, \alpha)$ $\quad$ $\beta(x) = 5$, $\beta(y) = 3$

$A = (\mathbb{N}, \alpha)$ $\quad\quad\quad \beta(x) = 5, \beta(y) = 3$

$$I \models plus(x,y) \equiv \underbrace{succ(succ(\dots succ}_{8 \text{ times}}(\sigma)\dots))$$

$$A \not\models plus(x,y) \equiv \underbrace{succ(\dots succ}_{8 \text{ times}}(\sigma)\dots)$$

$$A \models plus(succ(x),y) \equiv plus(x, succ(y))$$

<u>Def 223 (cont.)</u>

A set of equations $\mathcal{E}$ <u>entails</u> the equation $t_1 \equiv t_2$ (denoted $\mathcal{E} \models t_1 \equiv t_2$) iff for all algebras A with $A \models \mathcal{E}$ we also have $A \models t_1 \equiv t_2$.

Instead of $\emptyset \models t_1 \equiv t_2$ we also write $\models t_1 \equiv t_2$.
(means: every algebra A must be a model of $t_1 \equiv t_2$, i.e., $t_1 \equiv t_2$ is <u>valid</u>).

We define the relation $\equiv_{\mathcal{E}}$ on $\mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$ as:

$s \equiv_{\mathcal{E}} t$ iff $\mathcal{E} \models s \equiv t$.

<u>Ex. 2.2.4 (cont.):</u>

$$\{plus(x,y) \equiv plus(y,x)\} \models plus(\sigma, succ(\sigma)) \equiv plus(succ(\sigma), \sigma)$$

So if $\mathcal{E} = \{plus(x,y) \equiv plus(y,x)\}$, then

$$plus(\sigma, succ(\sigma)) \equiv_{\mathcal{E}} plus(succ(\sigma), \sigma)$$

<u>Ex 225</u>: $\mathcal{E}$ = set of the 3 group axioms from Ex. 2.1.7.

$A = (\mathbb{Z}, \alpha)$ with $\quad\quad A \models \mathcal{E}$

$\alpha_f(n,m) = n + m$

$\alpha_e = 0$

$\alpha_i(n) = -n$

One is interested in questions like:

"Does $i(i(n)) \equiv_{\mathcal{E}} n$ hold?"

(i.e.: $\mathcal{E} \models i(i(n))$?

Does $i(i(n)) \equiv n$ hold for every group?)

Word Problem: Given a set of equations $\mathcal{E}$ and 2 terms s and t, does $s \equiv_{\mathcal{E}} t$ hold?